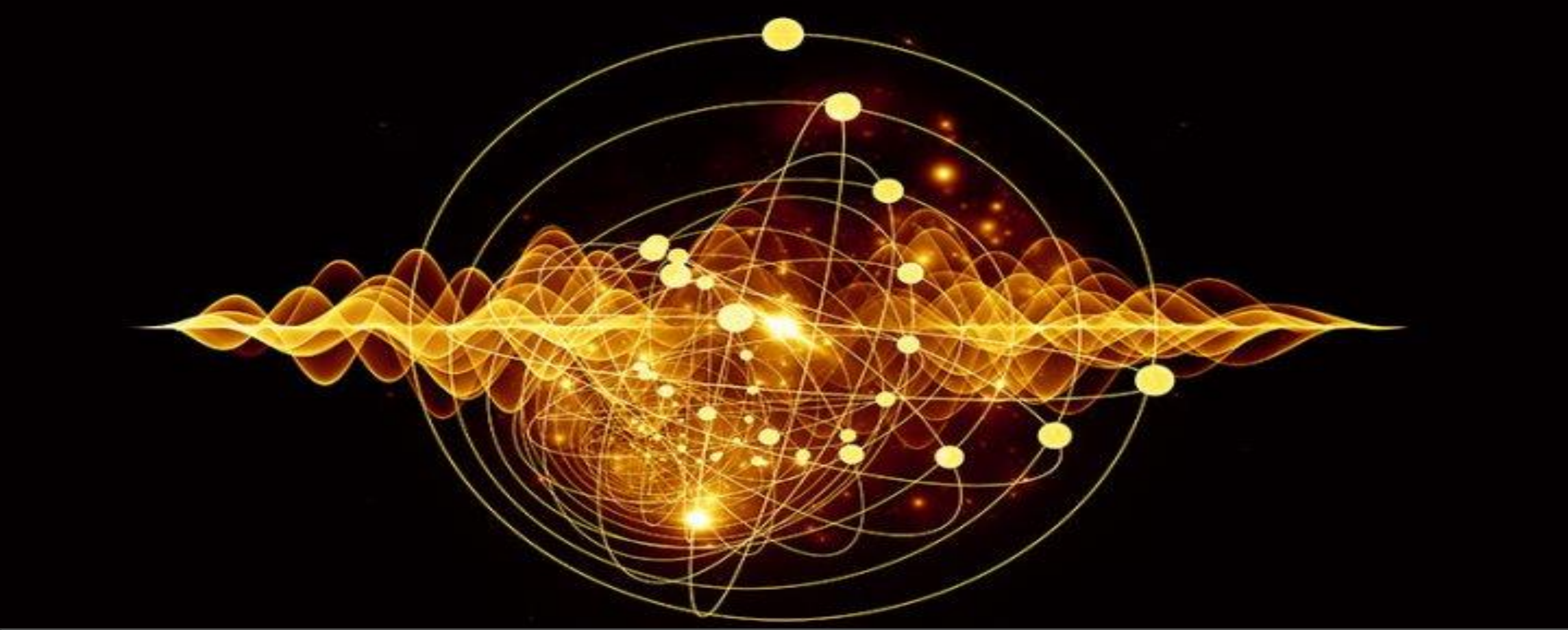


Particle Physics I

Lecture 7: The Dirac equation continued

Prof. Radoslav Marchevski
October 25th 2025

Recap and learning targets

- **Ultimate goal:** make predictions of particle decay rates and cross sections of particle scattering and compare the experimental results with the theoretical predictions

- Procedure: use Fermi`s golden rule: $\Gamma_{fi} = 2\pi\left|T_{fi}\right|^2\rho\left(E_f\right)$

  ✓ derive the expression for the density of final states (phase space) $\rho\left(E_f\right)$

  x calculate the matrix elements: our next target (this semester we will focus on quantum electrodynamics)

- **Today's learning targets**
  - solve Dirac's equation to find explicit forms of the wavefunctions of spin-half particles
  - spin and helicity operators
  - some fundamental symmetries: parity, charge conjugation, time reversal

Dirac equation and the properties of the γ matrices

- The Dirac equation can be written more elegantly by introducing the four Dirac gamma matrices

$$\gamma^0 = \beta, \qquad \gamma^1 = \beta\alpha_x, \qquad \gamma^2 = \beta\alpha_y, \qquad \gamma^3 = \beta\alpha_z \tag{1}$$

- Using $\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ we can rewrite it as

$$\left(i\gamma^\mu \partial_\mu - m\right)\Psi = 0 \tag{2}$$

- Properties of the α and β matrices: $\alpha_x = \alpha_x^\dagger, \alpha_y = \alpha_y^\dagger, \alpha_z = \alpha_z^\dagger, \beta = \beta^\dagger$

- For the γ matrices the full set of relations is:

$$(\gamma^0)^2 = I$$

$$(\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1$$

$$\gamma^0\gamma^j + \gamma^j\gamma^0 = 0$$

$$\gamma^k\gamma^j + \gamma^j\gamma^k = 0 \; (k \neq j)$$

- Which can be expresses as the anti-commutation rule (Clifford algebra):

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

Properties of the γ matrices

- Are the γ matrices Hermitian?
  - the β matrix is Hermitian $\Longrightarrow \gamma^0$ is also Hermitian
  - the α matrices are Hermitian giving:

$$\gamma^{1\dagger} = (\beta\alpha_x)^\dagger = \alpha_x^\dagger\beta^\dagger = \alpha_x\beta = -\beta\alpha_x = -\gamma^1$$

- From which follows that $\gamma^i (i = 1, 2, 3)$ are anti-Hermitian
- In summary:

$$\gamma^{0\dagger} = \gamma^0, \gamma^{1\dagger} = -\gamma^1, \gamma^{2\dagger} = -\gamma^2, \gamma^{3\dagger} = -\gamma^3$$

- Which can be expresses using a four-vector notation as

$$(\gamma^\mu)^\dagger = \gamma^0\,\gamma^\mu\,\gamma^0$$

Pauli-Dirac representation

- A possible numerical form of the γ −matrices:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

- Which written in full are:

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

Four-vector current and adjoint spinor

- Using the γ matrices $\rho = \Psi^\dagger\Psi$ and $\vec{j} = \Psi^\dagger\vec{\alpha}\Psi$ can be written as a four-vector current:

$$j^\mu = (\rho, \vec{j}) = \Psi^\dagger\gamma^0\gamma^\mu\Psi$$

- The continuity equation can be written in a Lorentz-invariant form of a 4-vector scalar product:

$$\partial_\mu j^\mu = 0$$

- The expression for the four-vector current $j^\mu = \Psi^\dagger\gamma^0\gamma^\mu\Psi$ can be simplified by introducing the **adjoint spinor $\overline{\Psi}$**

$$\overline{\Psi} = \Psi^\dagger\gamma^0$$

Four-vector current and adjoint spinor

$$\bar{\Psi} = \Psi^{\dagger}\gamma^{0}$$

$$\Downarrow$$

$$\bar{\Psi} = \Psi^{\dagger}\gamma^{0} = (\Psi^{*})^{T}\gamma^{0} = (\Psi_1^{*}, \Psi_2^{*}, \Psi_3^{*}, \Psi_4^{*})\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\Downarrow$$

$$\bar{\Psi} = (\Psi_1^{*}, \Psi_2^{*}, -\Psi_3^{*}, -\Psi_4^{*})$$

- In terms of the adjoint spinor the four-vector current can be written as

$$j^{\mu} = \bar{\Psi}\gamma^{\mu}\Psi$$

- And the adjoint (covariant) Dirac equation becomes

$$\left(i\partial_{\mu}\bar{\Psi}\gamma^{\mu} + m\bar{\Psi}\right) = 0 \Longrightarrow i\partial_{\mu}\bar{\Psi}\gamma^{\mu} = -m\bar{\Psi}$$

The Dirac equation: solution for a free particle at rest

- For a free particle ($V = 0$) at rest ($\vec{p} = 0$) we obtained four solutions:

$$\Psi_0^{(1)} = N\begin{pmatrix}1\\0\\0\\0\end{pmatrix}e^{-imt}, \qquad \Psi_0^{(2)} = N\begin{pmatrix}0\\1\\0\\0\end{pmatrix}e^{-imt}, \text{with positive energy}$$

$$\Psi_0^{(3)} = N\begin{pmatrix}0\\0\\1\\0\end{pmatrix}e^{+imt}, \qquad \Psi_0^{(4)} = N\begin{pmatrix}0\\0\\0\\1\end{pmatrix}e^{+imt}, \text{with negative energy}$$

- Four solutions: two with positive energy ($E > 0$) and two with negative energy ($E < 0$)

The Dirac equation: solution for a moving particle

$$(\gamma^\mu p_\mu - m)\Psi = 0$$

- We are looking for the solutions for a free particle with four-momentum $p^\mu = (E, \vec{p})$ in the form

  $\Psi = u(E, \vec{p}) e^{i(\vec{p}\cdot\vec{x} - Et)}$

$$\gamma^\mu p_\mu - m = \gamma^0 E - \vec{\gamma}\cdot\vec{p} - m$$

$$= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} E - \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \cdot \vec{p} - m \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} (E-m)I & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -(E+m)I \end{pmatrix}$$

- We can write the four-component spinor $u(E, \vec{p})$ as $u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$

$$(\gamma^\mu p_\mu - m)\Psi = 0 \Longrightarrow \begin{pmatrix} (E-m)I & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -(E+m)I \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The Dirac equation: solution for a moving particle

- We get two coupled simultaneous equations for u_A and u_B

$$\begin{cases} (\vec{\sigma}\cdot\vec{p})u_B = (E-m)u_A & (3) \\ (\vec{\sigma}\cdot\vec{p})u_A = (E+m)u_B & (4) \end{cases}$$

- Using the explicit form of the Pauli matrices we get

$$\vec{\sigma}\cdot\vec{p} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_x - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} p_y + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p_z$$

$$= \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} \qquad (5)$$

- From Eq.4 and Eq. 5 we get

$$u_B = \frac{\vec{\sigma}\cdot\vec{p}}{E+m} u_A = \frac{1}{E+m} \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} u_A$$

The Dirac equation: solution for a moving particle

- Solutions can be obtained by making the arbitrary (but simplest) choices for u_A:

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Longrightarrow u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \dfrac{p_z}{E+m} \\ \dfrac{p_x + ip_y}{E+m} \end{pmatrix} \tag{6}$$

normalisation factors

$$u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Longrightarrow u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \dfrac{p_x - ip_y}{E+m} \\ \dfrac{-p_z}{E+m} \end{pmatrix} \tag{7}$$

- Note: for $\vec{p} = 0$ we get the $E > 0$ particle-at-rest solutions
- The choice of u_A is arbitrary but that`s not an issue because we can express any other solution choice as a linear combination

The Dirac equation: solution for a moving particle

- Repeating the same procedure for u_B:

$$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Longrightarrow u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x + ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \qquad (8)$$

normalisation factors

$$u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Longrightarrow u_4 = N_4 \begin{pmatrix} \frac{p_x - ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix} \qquad (9)$$

- If any of these solutions is put back into the Dirac equation, we obtain $E^2 = \vec{p}^2 + m^2$, which doesn`t in itself identify the negative solutions

The Dirac equation: negative energy solutions

- It's not possible to interpret all four solutions as positive energy solutions
  - if we take all solutions to have the same value of E: $E = +|E|$ only two of the solutions are independent
  - there are only four independent solutions when two are taken to have $E < 0$
- To identify which solutions have $E < 0$ we can refer back to particle at rest, for $\vec{p} = 0$
  - u_1 and u_2 correspond to the $E > 0$ particle at rest solution
  - u_3 and u_4 correspond to the $E < 0$ particle at rest solution
- So u_1, u_2 are the positive energy solutions and u_3, u_4 the negative energy solutions

The Dirac equation: antiparticles

- Feynman-Stücklenberg interpretation:
  - **Approach 1:** negative energy solution interpreted as a negative energy particle propagating backwards in time
  - the time dependence of the wavefunction becomes: $e^{-iEt} = e^{-i(-E)(-t)}$
  - **Approach 2:** alternatively, it can be interpreted as a positive energy antiparticle propagating forward in time
- Following this interpretation we will work with antiparticle wavefunctions with $E = \sqrt{|\vec{p}|^2 + m^2}$

The Dirac equation: Approach 1

- **Approach 1:** start from the negative energy solutions

$$u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x + ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix}, \qquad u_4 = N_4 \begin{pmatrix} \frac{p_x - ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

- "Define" antiparticle wavefunction by flipping the sign of E and $\vec{p}$ and with E now being positive $E = \sqrt{|\vec{p}|^2 + m^2}$

$$v_1(E,\vec{p})e^{-i(\vec{p}\cdot\vec{x}-Et)} = u_4(-E,-\vec{p})e^{i(\vec{p}\cdot\vec{x}-Et)}$$

$$v_2(E,\vec{p})e^{-i(\vec{p}\cdot\vec{x}-Et)} = u_3(-E,-\vec{p})e^{i(\vec{p}\cdot\vec{x}-Et)}$$

The Dirac equation: Approach 2

- Approach 2: find negative energy plane-wave solutions to the Dirac equation of the form

  $\Psi = v(E,\vec{p})e^{-i(\vec{p}\cdot\vec{x}-Et)}$, where $E = \sqrt{|\vec{p}|^2 + m^2}$

- Although $E > 0$ these are still negative energy solutions: $\hat{H}\Psi = i\partial_t\Psi = -E\Psi$

- Putting Ψ in the Dirac equation: $(i\gamma^\mu\partial_\mu - m)\Psi = 0$ we get

$$\left(-\gamma^0 E + \gamma^1 p_x + \gamma^2 p_y + \gamma^3 p_z - m\right)v = 0$$

$$\Rightarrow \left(\gamma^\mu p_\mu + m\right)v = 0$$

Dirac equation in terms of momentum for antiparticles

Reminder: $(\gamma^\mu p_\mu - m)u = 0$ was the solution for particles

The Dirac equation: Approach 2

- We again get two coupled simultaneous equations this time for v_A and v_B

$$\begin{cases} (\vec{\sigma}\cdot\vec{p})v_A = (E-m)v_B & (10) \\ (\vec{\sigma}\cdot\vec{p})v_B = (E+m)v_A & (11) \end{cases}$$

antiparticles $E > 0$ $$v_1 = N_1' \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}, \qquad v_2 = N_2' \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

$u_4(-E,-\vec{p}) = v_1(E,\vec{p})$
$u_3(-E,-\vec{p}) = v_2(E,\vec{p})$

particles $E < 0$ $$u_4 = N_4 \begin{pmatrix} \frac{p_x - ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}, \qquad u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x + ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix}$$

The Dirac equation: particle and antiparticle spinors

- Four solution for "particles" of the form $\Psi_\mathrm{i} = u_i(E,\vec{p})e^{i(\vec{p}\cdot\vec{x}-Et)}$

$$u_1 = N_1\begin{pmatrix}1\\0\\\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\end{pmatrix}, u_2 = N_2\begin{pmatrix}0\\1\\\frac{p_x-ip_y}{E+m}\\\frac{-p_z}{E+m}\end{pmatrix}, u_3 = N_3\begin{pmatrix}\frac{p_z}{E-m}\\\frac{p_x+ip_y}{E-m}\\1\\0\end{pmatrix}, u_4 = N_4\begin{pmatrix}\frac{p_x-ip_y}{E-m}\\\frac{-p_z}{E-m}\\0\\1\end{pmatrix}$$

$u_1(-E,-\vec{p}) = v_3(E,\vec{p})$
$u_2(-E,-\vec{p}) = v_4(E,\vec{p})$

$E = \sqrt{|\vec{p}|^2+m^2}$

$E = -\sqrt{|\vec{p}|^2+m^2}$

- Four solution for "antiparticles" of the form $\Psi_i = v_i(E,\vec{p})e^{-i(\vec{p}\cdot\vec{x}-Et)}$

$$v_1 = N_1'\begin{pmatrix}\frac{p_x-ip_y}{E+m}\\\frac{-p_z}{E+m}\\0\\1\end{pmatrix}, v_2 = N_2'\begin{pmatrix}\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\\1\\0\end{pmatrix}, v_3 = N_3'\begin{pmatrix}1\\0\\\frac{p_z}{E-m}\\\frac{p_x+ip_y}{E-m}\end{pmatrix}, v_4 = N_4'\begin{pmatrix}0\\1\\\frac{p_x-ip_y}{E-m}\\\frac{-p_z}{E-m}\end{pmatrix}$$

$u_4(-E,-\vec{p}) = v_1(E,\vec{p})$
$u_3(-E,-\vec{p}) = v_2(E,\vec{p})$

$E = \sqrt{|\vec{p}|^2+m^2}$

$E = -\sqrt{|\vec{p}|^2+m^2}$

The Dirac equation: particle and antiparticle spinors

- We have a four-component spinor $\Longrightarrow$ only four are linearly independent
  - a natural choice is to use positive energy solutions: $\{u_1, u_2, v_1, v_2\}$

particles

antiparticles

$$u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix}, u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix},$$

$$v_1 = N_1' \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}, v_2 = N_2' \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

$$E = \sqrt{|\vec{p}|^2 + m^2}$$

$$E = \sqrt{|\vec{p}|^2 + m^2}$$

Normalisation and orthogonality

- The convention is to normalise the wavefunctions to $2|E|$ particles per unit volume
- For $\Psi = u_1 e^{i(\vec{p}\cdot\vec{x}-Et)}$ the probability density is $\rho = \Psi^\dagger\Psi = u_1^\dagger u_1 \;= |N_1|^2 \frac{2E}{E+m}$
- We are using only $E > 0$ solutions so we get for $\{u_1, u_2, v_1, v_2\}$:

$$N_1 = N_2 = N_1' = N_2' = N = \sqrt{E+m}$$

- The spinors are orthogonal

$$u_j^\dagger u_k = 0 \text{ for } j \neq k$$

$$u_j^\dagger u_k = 2|E|\delta_{jk} \text{ with } j, k = 1, 2, 3, 4$$

Short recap

- We solved the covariant Dirac equation for a free particle both at rest and in motion

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\Psi = 0$$

- We found four solutions for **particles** with 4-momentum $p^{\mu} = (E, \vec{p})$: $\Psi_i = u_i(E, \vec{p})e^{+i(\vec{p}\cdot\vec{x}-Et)}$
  - two solutions with $E > 0$ and two with $E < 0$
- We used the Feynman-Stücklenberg interpretation to interpret the negative solutions as positive energy **antiparticles** propagating forward in time: $\Psi_i = v_i(E, \vec{p})e^{-i(\vec{p}\cdot\vec{x}-Et)}$
  - two solutions with $E > 0$ and two with $E < 0$
- **8 solution in total, only 4 independent:** we chose to work with the $E > 0$ solutions $\{u_1, u_2, v_1, v_2\}$
- We normalised the solutions to $2|E|$ particles per unit volume giving

$$v_1(E, \vec{p}) = u_4(-E, -\vec{p})$$
$$v_2(E, \vec{p}) = u_3(-E, -\vec{p})$$

- **Orthogonal solutions:** $u_j^{\dagger}u_k = 2|E|\delta_{jk}$ with $j, k = 1, 2, 3, 4$

What about spin?

- Consider the orbital angular momentum operator $\vec{L} = \vec{x} \times \vec{p} = -i\vec{x} \times \vec{\nabla}$: does it commute with $\mathcal{H}_D$?

$$[\vec{L}, \mathcal{H}_D] = [\vec{x} \times \vec{p}, (\vec{\alpha} \cdot \vec{p} + \beta m)] = i\vec{\alpha} \times \vec{p} \neq 0 \Longrightarrow \text{angular momentum does not commute with } \mathcal{H}_\mathrm{D}$$

- The orbital angular momentum is NOT a conserved quantity of the quantum system

- Define the 4×4 operator $\vec{\Sigma}$ as an extension of the Pauli spin operator:

$$\vec{\Sigma} \text{ operator: } \vec{\Sigma} = (\Sigma_1, \Sigma_2, \Sigma_3) \equiv \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

$$\Sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \Sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Compute the commutator of $\vec{\Sigma}$ with $\mathcal{H}_D$

$$[\vec{\Sigma}, \mathcal{H}_D] = [\vec{\Sigma}, (\vec{\alpha} \cdot \vec{p} + \beta m)] = -2i(\vec{\alpha} \times \vec{p}) \neq 0 \Longrightarrow \text{spin also does not commute with } \mathcal{H}_\mathrm{D}$$

Spin of a Dirac particle

$$[\vec{L}, \mathcal{H}_D] = i\vec{\alpha} \times \vec{p} \text{ and } [\vec{\Sigma}, \mathcal{H}_D] = -2i(\vec{\alpha} \times \vec{p})$$

- We can define the **total angular momentum operator $\vec{J}$**

$$\vec{J} \equiv \vec{L} + \frac{1}{2}\vec{\Sigma} = \vec{L} + \vec{S}$$

- The quantity is conserved since its operator commutes with $\mathcal{H}_D$

$$[\vec{J}, \mathcal{H}_D] = [\vec{L} + \vec{S}, \mathcal{H}_D] = 0$$

Spin of a Dirac particle

- **The Dirac equation describes a relativistic particles with spin-1/2**
- The 4×4 matrix spin operator S is

$$\vec{S} = (S_1, S_2, S_3) = \frac{1}{2}\vec{\Sigma} = \frac{1}{2}\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

- The components of S have the same commutation relations as the Pauli matrices and of orbital angular momentum

$$[S_i, S_j] = 2i\epsilon_{ijk}S_k$$

- The spin magnitude of the Dirac particle is given by $\vec{S}^2\Psi = \mathrm{s}(\mathrm{s}+1)\Psi$ where

$$\vec{S}^2 = \frac{1}{4}(\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2) = \frac{3}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Spin of a Dirac particle: particle at rest

- Let`s consider the spinors for particles at rest $\Psi_0^i (i = 1, 2, 3, 4)$

$$\Psi_0^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt}, \qquad \Psi_0^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt}, \text{with positive energy}$$

$$\Psi_0^{(3)} = N \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{+imt}, \qquad \Psi_0^{(4)} = N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{+imt}, \text{with negative energy}$$

- They are eigenstates of the diagonal operator S_3:

$$S_3 = \frac{1}{2}\Sigma_3 = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- **Corresponding to spin-up $|\uparrow\rangle$ and spin-down $|\uparrow\rangle$ eigenstates**

Spin of a Dirac particle: moving particle

- Particle traveling along the z −direction, $p = (0,0,\pm p)$
- The solutions are given by
  - $\Psi^{(1,2)} = u_z^{(1,2)} e^{-ipx}$ for positive energy
  - $\Psi^{(3,4)} = u_z^{(3,4)} e^{+ipx}$ for negative energy

$$u_z^1 = N \begin{pmatrix} 1 \\ 0 \\ \frac{\pm p}{E+m} \\ 0 \end{pmatrix}, \qquad u_z^2 = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\mp p}{E+m} \end{pmatrix}, \qquad u_z^3 = N \begin{pmatrix} \frac{\pm p}{E-m} \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad u_z^4 = N \begin{pmatrix} 0 \\ \frac{\mp p}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

Spin of a Dirac particle: moving particle

- Particle traveling along the z −direction, $p = (0,0,\pm p)$

$$S_3\Psi^{(1)} = +\frac{1}{2}\Psi^{(1)}$$

$$S_3\Psi^{(2)} = -\frac{1}{2}\Psi^{(2)}$$

$$S_3\Psi^{(3)} = +\frac{1}{2}\Psi^{(3)}$$

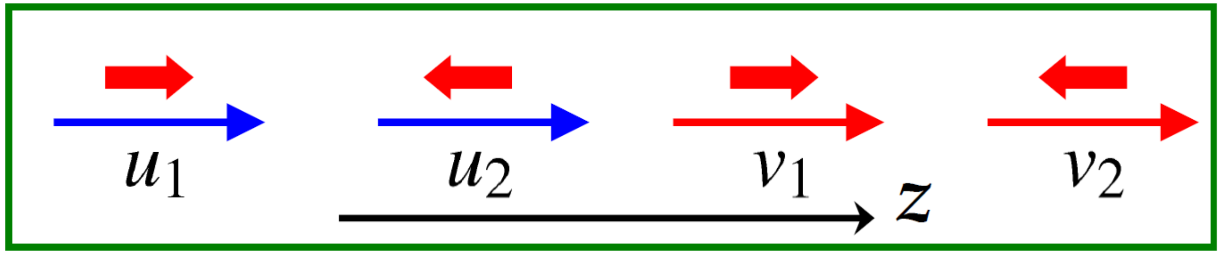
$$S_3\Psi^{(4)} = -\frac{1}{2}\Psi^{(4)}$$

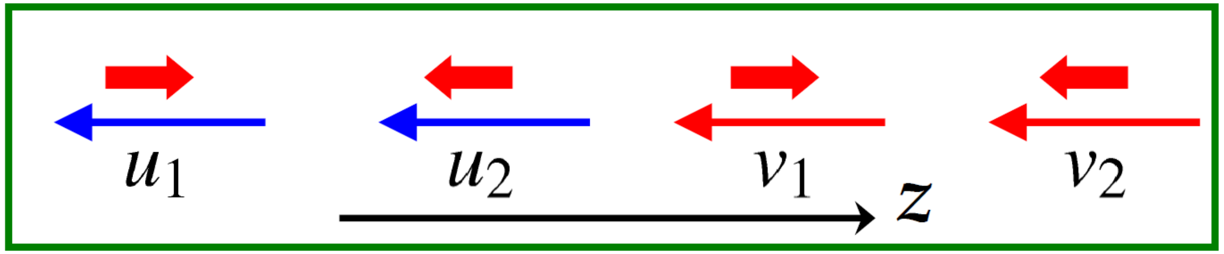
- Spinors $\Psi^{(1)}$ and $\Psi^{(3)}$ represent spin-up
- Spinors $\Psi^{(2)}$ and $\Psi^{(4)}$ represent spin-down

valid only for particles travelling along the z −direction

Spin states

- In general, the spinors $\{u_1, u_2, v_1, v_2\}$ are not eigenstates of S_3
- Only valid for particles and antiparticles traveling along the z −direction
- Can be represented as graphically for $(0,0,p)$ and $(0,0,-p)$





- More generally: we want to label our states in terms of “good quantum numbers”, i.e a set of observables commuting with $\mathcal{H}_D$, not only for particles that travel along the z −axis
- z −component of the spin would not work as $[\mathcal{H}_D, S_3] \neq 0$
- We must then introduce a new concept: **“helicity”**

Helicity of a Dirac particle

- The **helicity operator** represents the normalised projection of the spin along the direction of motion of the particle

$$h \equiv \frac{\vec{S}\cdot\vec{p}}{|\vec{S}||\vec{p}|} = \frac{2\vec{S}\cdot\vec{p}}{|\vec{p}|} = \frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}$$

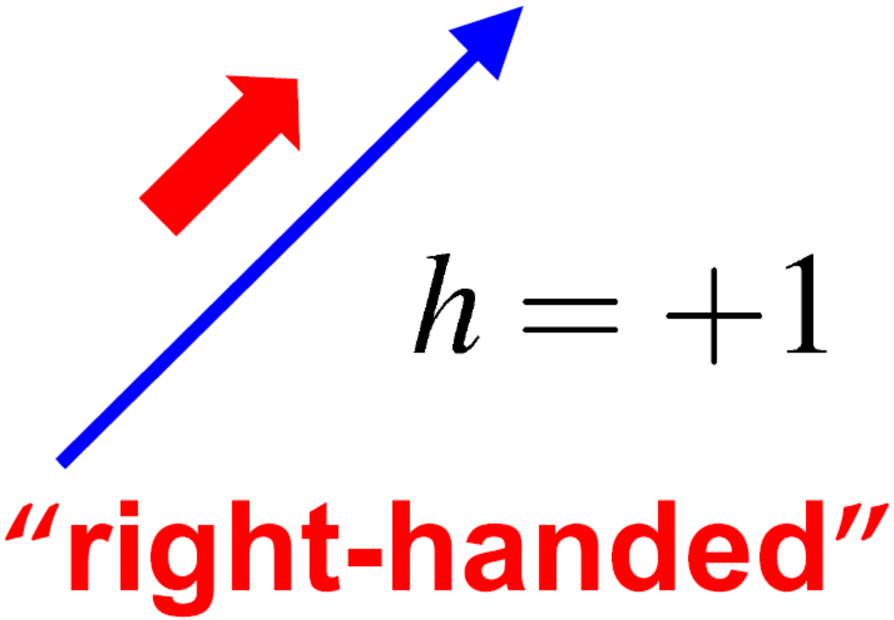
- The helicity operator commutes with $\mathcal{H}_D$ for a free particle ⟹ it is possible to define spinors that are simultaneous eigenstates of $\mathcal{H}_D$ and the helicity operator!

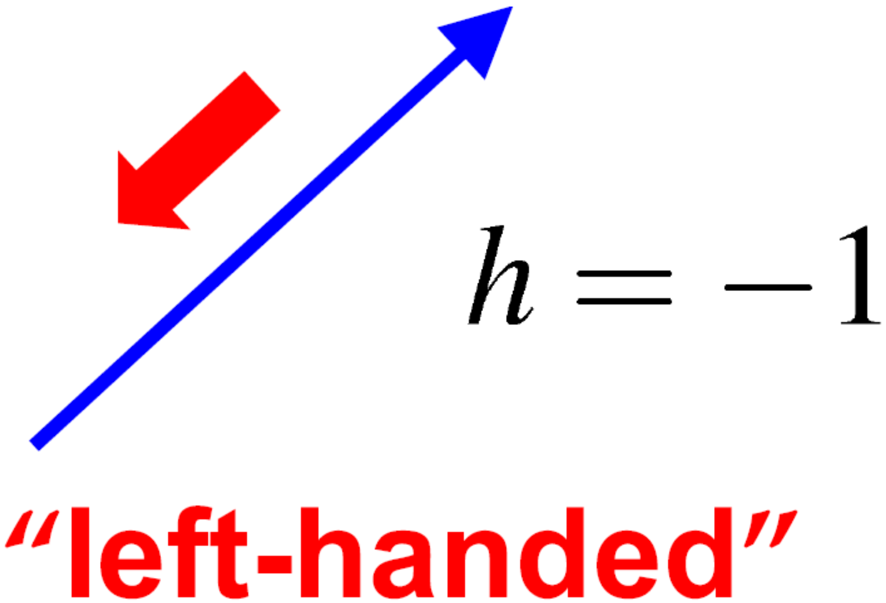
$$\left[\vec{\Sigma}\cdot\vec{p}, \mathcal{H}_D\right] = \left[\vec{\Sigma}\cdot\vec{p}, (\vec{\alpha}\cdot\vec{p} + \beta m)\right] = \left[\vec{\Sigma}\cdot\vec{p}, \vec{\alpha}\cdot\vec{p}\right] = 0, \text{since } \left[\vec{\Sigma}, \vec{\alpha}\right] = 0$$

- Note that $h^2 = \frac{1}{p^2}\begin{pmatrix} (\vec{\sigma}\cdot\vec{p})^2 & 0 \\ 0 & (\vec{\sigma}\cdot\vec{p})^2 \end{pmatrix} = \frac{1}{p^2}\begin{pmatrix} p^2 I & 0 \\ 0 & p^2 I \end{pmatrix} = I$
- $\Longrightarrow h = \pm 1$ and for a spin-1/2 particle the spin is quantized to be either "up" or "down"
- Helicity is a good quantum number with eigenvalues +1 and −1!

Helicity of a Dirac particle

- These states are called **positive** or **right-handed** and **negative** or **left-handed** helicity states





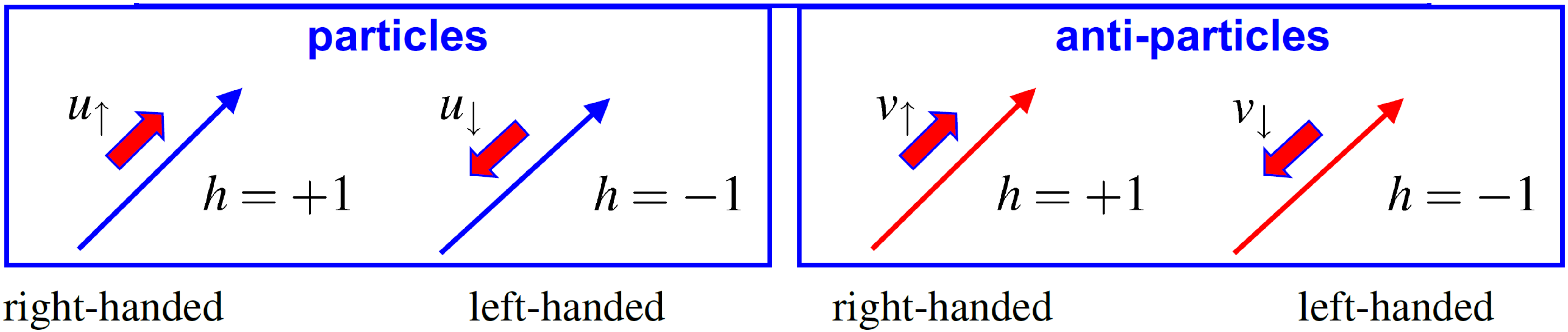
- If we make a measurement of the component of spin of a spin-1/2 particle along any axis it can take two values: $\pm 1/2$
- The eigenvalues of the helicity operator for a spin-1/2 particle are $h = \pm 1$

Helicity of a Dirac particle

- Note that even though it is a conserved quantity for a free Dirac particle, **helicity is not a Lorentz-invariant quantity**
- For any massive particle: $v < c$
- There exists a boosted inertial frame where the particle momentum appears reversed (not true for a massless particle travelling at the speed of light e.g. neutrinos)
- Relative to the boosted observer, the helicity of the particle will appear reversed
- The helicity is not invariant under Lorentz transformations (except for massless particles)

Helicity eigenstates

- See the complementary notes attached on moodle for a derivation of the helicity eigenstates
- Equivalent solutions and definition of right-handed and left-handed for antiparticles



Parity and time reversal

- Two discrete symmetries part of the Lorentz group

$$\text{Parity P:}\quad x^0 \to x^0; x^i \to -x^i$$

$$\text{Time reversal T:}\quad x^0 \to -x^0; x^i \to x^i (i = 1, 2, 3)$$

- Find a representation $S(P)$ and $S(T)$ of them on a Dirac spinor so that:

$$\Psi(t, \vec{x}) \to S(P)\Psi(t, -\vec{x}) \text{ and } \Psi(t, \vec{x}) \to S(T)\Psi(-t, \vec{x})$$

- Like rotations and boosts, the discrete transformations P and T should be representable by a 4×4 matrix, e.g. by the γ matrices

Parity reversal operator $\boldsymbol{P}$

- The operator P reverses the direction of the momentum $\vec{p}$ of a particle but it retains its spin
- The parity operator should satisfy: $P^{-1} = P$ and $P^2 = I$
- For simple handling of spin states, consider a particle moving along the z −direction (slide 26)
- The parity should not mix spin-up and spin-down configurations as well as positive and negative energy eigenstates
- The parity matches $u_z^{(i)}(E,\vec{p}) \leftrightarrow u_z^{(i)}(E,-\vec{p})$

$$S(P)u_z^{(1)}(E,-p) = N\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & ? & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & ? \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ \dfrac{-p}{E+m} \\ 0 \end{pmatrix} = N\begin{pmatrix} 1 \\ 0 \\ \dfrac{p}{E+m} \\ 0 \end{pmatrix} = u_z^{(1)}(E,p)$$

$\vec{p} = (0,0,p)$

$$S(P)u_z^{(2)}(E,-p) = N\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \dfrac{-p}{E+m} \end{pmatrix} = N\begin{pmatrix} 0 \\ 1 \\ 0 \\ \dfrac{p}{E+m} \end{pmatrix} = u_z^{(2)}(E,p)$$

Parity reversal operator $\boldsymbol{P}$

- The representation of the parity operator is γ^0

- The spinor representation of the parity operation is

$$\mathrm{P}: \Psi \rightarrow S(P)\Psi = \eta_P \gamma^0 \Psi$$

  - η_P is an overall unobservable phase

- For a particle/antiparticle at rest the solutions to the Dirac equations are

$$\Psi = u_1 e^{-imt}; \Psi = u_2 e^{-imt}; \Psi = v_1 e^{+imt}; \Psi = v_2 e^{+imt}$$

$$u_1 = N\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad u_2 = N\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad v_1 = N\begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \quad v_2 = N\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$$S(P)u_1 = \pm\begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & -1 & 0\\0 & 0 & 0 & -1\end{pmatrix}u_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \pm u_1$$

Parity reversal operator ***P***

- For all four spinors we get

$$S(P)u_1 = \pm u_1; \qquad S(P)v_1 = \mp v_1;$$

$$S(P)u_2 = \pm u_2; \qquad S(P)v_2 = \mp v_2;$$

- Hence the anti-particle at rest has opposite intrinsic parity to a particle at rest
- **Convention:** particles are chosen to have positive parity, which is equivalent to choosing

$$S(P) = +\gamma^0$$

What are scalar, vector, etc. particles

- The names come from how objects transform under parity reversal:
  - **scalar** is a constant: $P(s) = s$
  - **pseudoscalar** flips the sign: $P(p) = -p$
  - **vector** flips the sign: $P(\vec{v}) = -\vec{v}$
  - **pseudovector** or **axial vector** is unchanged: $P(\vec{a}) = \vec{a}$

- The same names are used for particles according to how they transform under parity
  - **scalar:** spin-0 particle with a positive parity 0^+, e.g. f_0 mesons (PDG)
  - **pseudoscalar:** spin-0 particle with a negative parity 0^-, e.g. pions $\pi^\pm, \pi^0$
  - **vector:** spin-1 particle with a negative parity 1^-, e.g. ρ, ω, γ, gluon
  - **pseudovector** or **axial vector:** spin-1 particle with a positive parity 1^+, e.g. f_1 mesons

Charge conjugation operator $\boldsymbol{C}$

- The operation that replaces each particle with its antiparticle and vice-versa keeping the spin unchanged is performed by the **charge conjugation operator $\boldsymbol{C}$**
- The charge conjugated spinor Ψ_C is defined as

Charge conjugation C: $\Psi \rightarrow \Psi_C = C\Psi^*$

- In analogy with the Schrödinger equation, the Dirac equation for a particle of charge e in an external electromagnetic field is:

$$\big(i\gamma^{\mu}(\partial_{\mu} - eA_{\mu}) - m\big)\Psi = 0 \qquad (12)$$

- For the charge conjugated particle (of charge $-e$) it should be

$$\big(i\gamma^{\mu}(\partial_{\mu} + eA_{\mu}) - m\big)\Psi = 0 \qquad (13)$$

- Ψ_C should satisfy the above equation

Charge conjugation operator $\boldsymbol{C}$

- Taking complex conjugate of (12) and multiplying by C we get

$$C\big(-i(\gamma^\mu)^*(\partial_\mu - eA_\mu) - m\big)C^{-1}C\Psi^* =$$
$$= \big(-iC(\gamma^\mu)^*C^{-1}(\partial_\mu - eA_\mu) - m\big)\Psi_\mathrm{C} = 0$$

- For Eq.13 to work we need

$$C(\gamma^\mu)^*C^{-1} = -\gamma^\mu$$

- In our representations, only γ^2 is imaginary: $C = i\gamma^2$ works

Charge conjugation C: $\quad \Psi \to \Psi_C = i\eta_C\gamma^2\Psi^*$

  - η_C is an unobservable global phase

Charge conjugation operator $\boldsymbol{C}$

- We verify directly the effect of the charge conjugation on our specific particle and antiparticle solutions using the spinors $u^{(i)}$ and $v^{(i)}$ of the Dirac equation

$$C v^{(1)}(E,\vec{p}) = i\gamma^2 v^{(1)*} = i\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} N \begin{pmatrix} \frac{p_x + ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix} = u^{(1)}(E,\vec{p})$$

$$C v^{(2)}(E,\vec{p}) = i\gamma^2 v^{(2)*} = i\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} N \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x - ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix} = -N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} = -u^{(2)}(E,\vec{p})$$

- The effect of the charge conjugation operator on the antiparticle spinors $v^{(1)}$ and $v^{(2)}$ is to transform them into $u^{(1)}$ and $u^{(2)}$ (up to an unobservable complex phase)

Time reversal operator $\boldsymbol{T}$

- Time reversal operator flips the momentum $\vec{p}$ and the spin
- Like previously, we look for an operator T such that

$$\text{Time reversal T: } \Psi(t,\vec{x}) \rightarrow S(T)\Psi(-t,\vec{x})$$

- The operator must be *antiunitary* satisfy $T^{-1} = -T$ and $T^2 = -I$
- Consider the product $i\gamma^1\gamma^3$ (and use $\sigma_1\sigma_3 = i\epsilon_{132}\sigma_2 = -i\sigma_2$):

$$S(T) = \eta_T i\gamma^1\gamma^3 = -\eta_T i \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} = \eta_T i \begin{pmatrix} -\sigma_1\sigma_3 & 0 \\ 0 & -\sigma_1\sigma_3 \end{pmatrix} = -\eta_T \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

  - where η_T is an overall unobservable phase

Time reversal operator $\boldsymbol{T}$

- Again, a particle moving along the z −direction

$$S(T)u_z^{(1)}(E,p) = \eta_T N \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{p}{E+m} \\ 0 \end{pmatrix} = -i\eta_T N \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{p}{E+m} \end{pmatrix} = -i\eta_T u_z^{(2)}(E,p)$$

$$S(T)u_z^{(2)}(E,p) = \eta_T N \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{p}{E+m} \end{pmatrix} = i\eta_T N \begin{pmatrix} 1 \\ 0 \\ \frac{p}{E+m} \\ 0 \end{pmatrix} = i\eta_T u_z^{(1)}(E,p)$$

- $\Longrightarrow$ the time reversal transformation changes spin and can be expressed as

Time reversal T : $\Psi \rightarrow S(T)\Psi = \eta_T i\gamma^1\gamma^3\Psi$

Summary

- We formulated a relativistic quantum mechanics starting from the linear Dirac equation

$$\mathcal{H}_D \Psi = (\vec{\alpha} \cdot \vec{p} + \beta m)\Psi = i\frac{\partial \Psi}{\partial t} \Longrightarrow \left(i\gamma^\mu \partial_\mu - m\right)\Psi = 0$$

  - new degrees of freedom: found to describe spin-1/2 particles

- We introduced the 4-vector current and adjoint spinor:

$$j^\mu = \Psi^\dagger \gamma^0 \gamma^\mu \Psi = \bar{\Psi}\gamma^\mu \Psi$$

- With the Dirac equation we can`t escape from having two $E > 0$ and two $E < 0$ solutions

- We used the Feynman-Stücklenberg interpretation
  - $E > 0$ solutions: positive energy **particles** propagating forward in time: $\Psi_i = u_i(E, \vec{p})e^{+i(\vec{p}\cdot\vec{x} - Et)}$
  - $E < 0$ solutions: positive energy **antiparticles** propagating forward in time: $\Psi_i = v_i(E, \vec{p})e^{-i(\vec{p}\cdot\vec{x} - Et)}$

- 8 solution in total, only 4 independent: we chose to work with the $E > 0$ solutions $\{u_1, u_2, v_1, v_2\}$

- Orthogonal solutions: $u_j^\dagger u_k = 2|E|\delta_{jk}$ with $j, k = 1, 2, 3, 4$

Summary

- The most useful basis: particle and antiparticle helicity eigenstates $\{u_1, u_2, v_1, v_2\}$
- In terms of the 4-component spinors, the charge conjugation, parity and time reversal operations are:

Charge conjugation C : $\Psi \to \Psi_C = i\eta_C \gamma^2 \Psi^*$

Parity reversal P : $\Psi \to S(P)\Psi = \eta_P \gamma^0 \Psi$

Time reversal T : $\Psi \to S(T)\Psi = \eta_T i\gamma^1 \gamma^3 \Psi$

Summary of Lecture 7

Main learning outcomes

- Dirac equation
    - 4-vector current and adjoint spinors
    - solutions in terms of particle spinors representing spin-1/2 particles
    - choosing the appropriate basis of spinors
- Spin and helicity operators
- Charge, Parity and Time reversal operators acting on spinors